2022-2023 Term 2 MATH 116

JC Wang

University of Saskatchewan

March 11, 2023
Appendix E Sigma Notation

Definition
A sequence is a set of objects ordered by positive integers. (These objects are usually numbers.) A sequence is said to be finite if it is finite as a set. A sequence is said to be infinite if it is not a finite sequence.

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Given a finite sequence \( a_m, a_{m+1}, \ldots, a_n \) (where \( m \) and \( n \) are positive integers with \( m \leq n \)), we use the following sigma notation for their sum:

\[
\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n.
\]

Example
\[
\sum_{i=5}^{7} 6 = 6 + 6 + 6, \quad \sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2.
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Rules of Summation

- Let $c$ be a constant that is independent of the index $i$. Then
  \[ \sum_{i=m}^{n} c = c \cdot \text{(the number of terms)} = c(n - m + 1). \]
- \[ \sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i. \]
- \[ \sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i. \]
- \[ \sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i. \]

Theorem

\[ \sum_{i=1}^{n} i = n(n + 1)/2. \]

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\[ \sum_{i=1}^{n} i^2 = n(n + 1)(2n + 1)/6. \]

Theorem

\[ \sum_{i=1}^{n} i^3 = [n(n + 1)/2]^2. \]

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\[ \sum_{i=1}^{n} x^i = x(1 - x^n)/(1 - x) \text{ for } x \neq 1. \]
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Example

Find the limit

\[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n} \left( \left( \frac{i}{n} \right)^2 + 1 \right). \]
5-1 The Area Problem

Example

Find the area under the curve \( y = x^2 \) from \( x = 0 \) to \( x = 1 \).

Definition

Let \( f \) be a **nonnegative**, **continuous** function on an interval \([a, b]\). Let

\[
\Delta x = \frac{b - a}{n}, \quad x_i = a + i\Delta x \quad \text{for} \; i = 0, 1, 2, \cdots, n.
\]

Choose a point \( x_i^* \) from the \( i \)-th closed subinterval \([x_{i-1}, x_i]\). Define the **Riemann sum** and the **area** under the curve \( y = f(x), \; a \leq x \leq b \), by

\[
\sum_{i=1}^{n} f(x_i^*)\Delta x = \begin{cases} 
\text{upper sum} & \text{if one chooses} \quad f(x_i^*) = \max_{x_{i-1} \leq x \leq x_i} f(x), \\
\text{lower sum} & \text{if one chooses} \quad f(x_i^*) = \min_{x_{i-1} \leq x \leq x_i} f(x).
\end{cases}
\]

The area = \( \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \).
5-1 The Area Problem

Example

Find the area under the curve $y = x^2$ from $x = 0$ to $x = 1$.

Definition

Let $f$ be a nonnegative, continuous function on an interval $[a, b]$. Let

$$
\Delta x = \frac{b - a}{n}, \quad x_i = a + i\Delta x \quad \text{for } i = 0, 1, 2, \cdots, n.
$$

Choose a point $x_i^*$ from the $i$-th closed subinterval $[x_{i-1}, x_i]$. Define the Riemann sum and the area under the curve $y = f(x)$, $a \leq x \leq b$, by

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The area $= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$. 
5-1 The Distance Problem

Definition

An object moves with \textbf{continuous} velocity $f(t)$, where $a \leq t \leq b$ and $f(t) \geq 0$. Let

$$\Delta t = \frac{b - a}{n}, \quad t_i = a + i\Delta t \quad \text{for } i = 0, 1, 2, \cdots, n.$$ 

Choose a point $t_i^*$ from the $i$-th closed subinterval $[t_{i-1}, t_i]$. Define the \textbf{distance} traveled during the time interval $[a, b]$ by

\[
\text{The distance} = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i^*)\Delta t.
\]
5-2 The Definite Integral

**Definition**

Let \( f \) be a **continuous** function on an interval \([a, b]\). Let

\[
\Delta x = \frac{b - a}{n}, \quad x_i = a + i\Delta x \quad \text{for } i = 0, 1, 2, \ldots, n.
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Choose a point \( x_i^* \) from the \( i \)-th closed subinterval \([x_{i-1}, x_i]\). Define the **definite integral** (or simply **integration** or **integral**) of \( f \) over \([a, b]\) by

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.
\]

- \( f \) not required to be positive; the sample point \( x_i^* \) can be arbitrary, for example, the mid-point \( x_i^* = (x_{i-1} + x_i)/2 \).
- Call \( f(x) \) the **integrand**; \( a, b \) the **limits of integration**.
- Could use any letter in place of \( x \) without changing the value of the integral.
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- When \( f \geq 0 \) on \([a, b]\),

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- When $f \leq 0$ on $[a, b]$,

$$\int_a^b f(x) \, dx = -\text{area under the curve } y = f(x), \ a \leq x \leq b.$$
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\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.
\]

Example

Express \( \lim_{n \to \infty} \sum_{i=1}^{n} \left( x_i^3 + x_i \sin x_i \right) \Delta x \) as an integral on the interval \([0, \pi]\).
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$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$ 

Example

Approximate the definite integral $\int_0^8 \sin \sqrt{x} \, dx$ using Riemann sums in the case of $n = 4$. 
5-2 The Definite Integral

- If \( c \) is a constant, then 
  \[
  \int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx \quad \text{and} \quad \int_a^b c \, dx = c(b - a)
  \]
- \[
  \int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx
  \]
- If \( a < c < b \) then 
  \[
  \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
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  \int_a^a f(x) \, dx = 0
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- When \( f \) takes both positive and negative values, then 
  \[
  \int_a^b f(x) \, dx = \text{the net area of } f \text{ over } [a, b].
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Example

Find the value of 
\[
\int_0^1 (\sqrt{1 - x^2} - 6x) \, dx.
\]
5-2 The Definite Integral

- $c$ constant $\Rightarrow \int_a^b cf(x)\,dx = c \int_a^b f(x)\,dx$ and $\int_a^b c\,dx = c(b-a)$

- $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

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Example

Find the value of

$$\int_0^1 (\sqrt{1 - x^2} - 6x) \, dx.$$
5-2 The Definite Integral

- \( f \geq 0 \) on \([a, b]\) \( \Rightarrow \) \( \int_a^b f(x) \, dx \geq 0 \)
- \( f \geq g \) on \([a, b]\) \( \Rightarrow \) \( \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \)
- \( m \leq f \leq M \) on \([a, b]\) \( \Rightarrow \) \( m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a) \)
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The Fundamental Theorem of Calculus (FTC)

Theorem

Assume that $f$ is continuous on $[a, b]$.

1. The function

$$g(x) = \int_a^x f(t) \, dt, \quad a \leq x \leq b,$$

is continuous on $[a, b]$ and differentiable on $(a, b)$, and

$$g'(x) = f(x), \quad a < x < b.$$

2. The value of the integral

$$\int_a^b f(x) \, dx = F(b) - F(a),$$

where $F$ is any anti-derivative of $f$, that is, a function $F$ such that $F' = f$. 

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Examples

Example
Find the derivative of
\[ \int_0^x \sqrt{1 + t^2} \, dt. \]

Example
Find the derivative of
\[ \int_0^{x^4} \sec t \, dt. \]

Example
Evaluate
\[ \int_3^6 \frac{1}{x} \, dx. \]
Examples

Example
What’s wrong with the calculation?

\[
\int_{-1}^{3} \frac{1}{x^2} \, dx = -\frac{1}{x} \bigg|_{x=-1}^{x=3} = -\frac{1}{3} - 1 < 0.
\]

Example
Find the limit

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n} \left[ \left( \frac{i}{n} \right)^2 + 1 \right].
\]

Example
Find the limit

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \left[ \left( \frac{2i}{n} \right)^3 + 5 \left( \frac{2i}{n} \right) \right].
\]
The notation \( F(x) = \int f(x) \, dx \) means \( F'(x) = f(x) \).

Example

\[
\int x^2 \, dx = \frac{x^3}{3} + \text{constant}
\]

\[
\int \sin x \, dx = -\cos x + \text{constant}
\]
5-4 The Indefinite Integral

Definition

The notation

\[ F(x) = \int f(x) \, dx \]

means

\[ F'(x) = f(x). \]

Example

\[ \int x^2 \, dx = \frac{x^3}{3} + \text{constant} \]

\[ \int \sin x \, dx = -\cos x + \text{constant} \]
5-4 The Indefinite Integral

Examples
(The Table of Indefinite Integrals Part I)

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + \text{constant} \quad (n \neq -1) \]

\[ \int \frac{1}{x} \, dx = \ln |x| + \text{constant} \]

\[ \int e^x \, dx = e^x + \text{constant}, \quad \int b^x \, dx = \frac{b^x}{\ln b} + \text{constant} \]
5-4 The Indefinite Integral

Examples

(The Table of Indefinite Integrals Part II)

\[
\int \sin x \, dx = -\cos x + \text{constant}, \quad \int \cos x \, dx = \sin x + \text{constant}
\]

\[
\int \sec^2 x \, dx = \tan x + \text{constant}, \quad \int \csc^2 x \, dx = -\cot x + \text{constant}
\]

\[
\int \sec x \tan x \, dx = \sec x + \text{constant}, \quad \int \csc x \cot x \, dx = -\csc x + \text{constant}
\]

\[
\int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + \text{constant}, \quad \int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + \text{constant}
\]
Example

Define

\[
\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}.
\]

Then we have

\[
\int \sinh x \, dx = \cosh x + \text{constant}, \quad \int \cosh x \, dx = \sinh x + \text{constant}.
\]

Example

Compute

\[
\int \frac{\cos \theta}{\sin^2 \theta} \, d\theta.
\]

Example

Evaluate

\[
\int_0^2 \left(2x^3 - 6x + \frac{3}{1 + x^2}\right) \, dx.
\]
Example

Define
\[ \sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}. \]

Then we have
\[ \int \sinh x \, dx = \cosh x + \text{constant}, \quad \int \cosh x \, dx = \sinh x + \text{constant}. \]

Example

Compute
\[ \int \frac{\cos \theta}{\sin^2 \theta} \, d\theta. \]

Example

Evaluate
\[ \int_0^2 2x^3 - 6x + \frac{3}{1 + x^2} \, dx. \]
5-4 The Indefinite Integral

Example

Define

\[
\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}.
\]

Then we have

\[
\int \sinh x \, dx = \cosh x + \text{constant}, \quad \int \cosh x \, dx = \sinh x + \text{constant}.
\]

Example

Compute

\[
\int \frac{\cos \theta}{\sin^2 \theta} \, d\theta.
\]

Example

Evaluate

\[
\int_0^2 2x^3 - 6x + \frac{3}{1 + x^2} \, dx.
\]
Example

Evaluate

\[ \int_{1}^{9} \frac{2t^2 + t^2 \sqrt{t} - 1}{t^2} \, dt. \]
(Net Change Theorem) If \( F \) is differentiable on some open interval that contains \([a, b]\), then

\[
\int_a^b F'(x) \, dx = F(b) - F(a).
\]

This is a reformulation of the FTC.

An object moves alone the real line with position \( s(t) \), then its velocity is \( v(t) = s'(t) \), so

**displacement** during the time period \([t_1, t_2]\) = \( \int_{t_1}^{t_2} v(t) \, dt = s(t_2) - s(t_1) \);

**distance** traveled during the time period \([t_1, t_2]\) = \( \int_{t_1}^{t_2} |v(t)| \, dt \).
5-4 The Indefinite Integral: Applications

Theorem

(Net Change Theorem) If $F$ is differentiable on some open interval that contains $[a, b]$, then

$$\int_{a}^{b} F'(x) \, dx = F(b) - F(a).$$

This is a reformulation of the FTC.

Example

An object moves alone the real line with position $s(t)$, then its velocity is $v(t) = s'(t)$, so

**displacement** during the time period $[t_1, t_2] = \int_{t_1}^{t_2} v(t) \, dt = s(t_2) - s(t_1)$;

**distance** traveled during the time period $[t_1, t_2] = \int_{t_1}^{t_2} |v(t)| \, dt$. 
Example

A particle moves on the real line with \( v(t) = t^2 - t - 6 \).

1. Find the displacement of the particle during the time period \( 1 \leq t \leq 4 \).
2. Find the distance traveled during this time period.
5-5 The Substitution Rule

Example

\[ \int 2x \sqrt{1 + x^2} \, dx \]

Fact

(The Substitution Rule) If \( u = g(x) \) is differentiable and its range is an interval \( I \) on which \( f \) is continuous, then

\[ \int f(g(x))g'(x) \, dx = \int f(u) \, du. \]

Example

\[ \int x^3 \cos(x^4 + 2) \, dx \]

Example

\[ \int \sqrt{2x + 1} \, dx \]
5-5 The Substitution Rule

Example
\[ \int 2x \sqrt{1 + x^2} \, dx \]

Fact
(The Substitution Rule) If \( u = g(x) \) is differentiable and its range is an interval \( I \) on which \( f \) is continuous, then

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Example
\[ \int x^3 \cos(x^4 + 2) \, dx \]

Example
\[ \int \sqrt{2x + 1} \, dx \]
### 5-5 The Substitution Rule

**Example**

\[ \int \frac{x}{\sqrt{1-4x^2}} \, dx \]

**Example**

\[ \int e^{5x} \, dx \]

**Example**

\[ \int x^5 \sqrt{x^2+1} \, dx; \quad u = x^2 + 1 \]

**Example**

\[ \int \tan x \, dx \]

**Example**

\[ \int_1^2 \frac{1}{(3-5x)^2} \, dx \]
5-5 The Substitution Rule

Example
\[ \int_1^e \frac{\ln x}{x} \, dx \]

Fact
(Symmetry) Suppose \( f \) is continuous on \([-a, a]\).

1. If \( f(-x) = f(x) \) for all \( x \), then \( \int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx \).
2. If \( f(-x) = -f(x) \) for all \( x \), then \( \int_{-a}^a f(x) \, dx = 0 \).

Example
\[ \int_{-2}^2 x^6 + 1 \, dx \]

Example
\[ \int_{-1}^1 \frac{\tan x}{1+x^2+x^4} \, dx = 0. \]
5-5 The Substitution Rule

Example
\[ \int_1^e \frac{\ln x}{x} \, dx \]

Fact

(Symmetry) Suppose \( f \) is continuous on \([-a, a]\).

1. If \( f(-x) = f(x) \) for all \( x \), then \( \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx \).
2. If \( f(-x) = -f(x) \) for all \( x \), then \( \int_{-a}^{a} f(x) \, dx = 0 \).

Example
\[ \int_{-2}^{2} x^6 + 1 \, dx \]

Example
\[ \int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4} \, dx = 0. \]
6-1 The Area Between Curves

Fact

The area bounded by the continuous curves \( y = f(x), \ y = g(x), \) and the lines \( x = a, \ x = b \) is given by

\[
\text{area} = \int_a^b |f(x) - g(x)| \, dx.
\]

Example

Find the area of the region bounded by \( y = e^x, \ y = x, \ x = 0, \ x = 1. \)

Example

Find the area enclosed by parabolas \( y = x^2 \) and \( y = 2x - x^2. \)

Example

Find the area bounded by \( y = \sin x, \ y = \cos x, \ x = 0, \ x = \pi/2. \)
6-1 The Area Between Curves

Fact

The area bounded by the continuous curves \( y = f(x) \), \( y = g(x) \), and the lines \( x = a \), \( x = b \) is given by

\[
\text{area} = \int_a^b |f(x) - g(x)| \, dx.
\]

Example

Find the area of the region bounded by \( y = e^x \), \( y = x \), \( x = 0 \), \( x = 1 \).

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Find the area enclosed by parabolas \( y = x^2 \) and \( y = 2x - x^2 \).

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Find the area bounded by \( y = \sin x \), \( y = \cos x \), \( x = 0 \), \( x = \pi/2 \).
6-1 The Area Between Curves

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The area bounded by the continuous curves \( y = f(x), \ y = g(x), \) and the lines \( x = a, \ x = b \) is given by

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Example

Find the area of the region bounded by \( y = e^x, \ y = x, \ x = 0, \ x = 1. \)

Example

Find the area enclosed by parabolas \( y = x^2 \) and \( y = 2x - x^2. \)

Example

Find the area bounded by \( y = \sin x, \ y = \cos x, \ x = 0, \ x = \pi/2. \)
Some regions are best treated by regarding $x$ as a function in $y$. The area bounded by the continuous curves $x = f(y)$, $x = g(y)$, and the lines $y = c$, $y = d$ is given by

$$\text{area} = \int_{c}^{d} |f(y) - g(y)| \, dy.$$
Let $S$ be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of $S$ through $x$ and perpendicular to the $x$-axis is a continuous function $A(x)$, then

$$\text{volume of } S = \int_a^b A(x) \, dx.$$ 

Example

Find the volume of a ball with radius $r$.

$$A(x) = \pi y^2 = \pi (r^2 - x^2), \quad -r \leq x \leq r.$$
6-2 Volumes

Fact

Let $S$ be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of $S$ through $x$ and perpendicular to the $x$-axis is a continuous function $A(x)$, then

$$\text{volume of } S = \int_a^b A(x) \, dx.$$  

Example

Find the volume of the solid obtained by rotating about the $x$-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

$$A(x) = \pi (\sqrt{x})^2, \quad 0 \leq x \leq 1.$$
Fact

Let $S$ be a solid that lies between $y = c$ and $y = d$. If the cross-sectional area of $S$ through $y$ and perpendicular to the $y$-axis is a continuous function $A(y)$, then

$$\text{volume of } S = \int_{c}^{d} A(y) \, dy.$$  

Example

Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about $y$-axis.

$$A(y) = \pi x^2 = \pi (\sqrt[3]{y})^2, \quad 0 \leq y \leq 8.$$
Example

The region $R$ enclosed by $y = x$ and $y = x^2$ is rotated about the $x$-axis. Find the volume of the resulting solid.

$$A(x) = \pi x^2 - \pi (x^2)^2, \quad 0 \leq x \leq 1.$$ 

Example

Rotate the same region $R$ about the horizontal line $y = 2$ and find the volume of the solid of revolution whose cross-section is a washer with the inner radius $2 - x$ and the outer radius $2 - x^2$. 

Example

The region $R$ enclosed by $y = x$ and $y = x^2$ is rotated about the $x$-axis. Find the volume of the resulting solid.

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Example

Rotate the same region $R$ about the horizontal line $y = 2$ and find the volume of the solid of revolution whose cross-section is a washer with the inner radius $2 - x$ and the outer radius $2 - x^2$. 

The same region $R$ enclosed by $y = x$ and $y = x^2$ is now rotated about the vertical line $x = -1$. Find the volume of the solid of revolution whose cross-section is now a washer with the inner radius $1 + y$ and the outer radius $1 + \sqrt{y}$.
Example

A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder, while the other intersects the first at an angle of $30^\circ$ along a diameter of the cylinder. Find the volume of the wedge.

Hint: Place the $x$-axis along the diameter where the planes meet, then the base of the solid is a semicircle $y = \sqrt{16 - x^2}$, $-4 \leq x \leq 4$. Then the cross-section perpendicular to the $x$-axis at $x$ is a triangle whose base is $y = \sqrt{16 - x^2}$ and the height $y \tan 30^\circ$. Thus,

$$A(x) = \frac{16 - x^2}{2\sqrt{3}}, \quad -4 \leq x \leq 4.$$
6-3 Volumes by Cylindrical Shells

**Fact**

Let $S$ be the solid obtained by rotating about the $y$-axis the region bounded by $y = f(x)$ where $f(x) \geq 0$, $y = 0$, $x = a$, $x = b$, where $b > a \geq 0$. Then

$$\text{volume of } S = \int_{a}^{b} 2\pi x f(x) \, dx.$$  

**Example**

Find the volume of $S$ where the region is bounded by $y = f(x) = 2x^2 - x^3$ and $y = 0$.

$$\text{volume of } S = \int_{a}^{b} 2\pi x f(x) \, dx.$$
6-3 Volumes by Cylindrical Shells

Fact

Let $S$ be the solid obtained by rotating about the $y$-axis the region bounded by $y = f(x)$ where $f(x) \geq 0$, $y = 0$, $x = a$, $x = b$, where $b > a \geq 0$. Then

$$\text{volume of } S = \int_{a}^{b} 2\pi xf(x) \, dx.$$ 

Example

Find the volume of $S$ where the region is bounded by $y = f(x) = 2x^2 - x^3$ and $y = 0$.

$$\text{volume of } S = \int_{a}^{b} 2\pi x \left( f(x) \right) \, dx.$$ 

\[ \text{circumference} \quad \text{height} \quad \text{thickness} \]
6-3 Volumes by Cylindrical Shells

**Fact**

Let $S$ be the solid obtained by rotating about the $y$-axis the region bounded by $y = f(x)$ where $f(x) \geq 0$, $y = 0$, $x = a$, $x = b$, where $b > a \geq 0$. Then

$$\text{volume of } S = \int_a^b 2\pi xf(x) \, dx.$$ 

**Example**

Find the volume of the solid obtained by rotating about the $y$-axis the region between $y = x$ and $y = x^2$.

height $f(x) = x - x^2$
Fact

Let $S$ be the solid obtained by rotating about the $x$-axis the region bounded by $x = g(y)$ where $g(y) \geq 0$, $x = 0$, $y = c$, $y = d$, where $d > c \geq 0$. Then

$$\text{volume of } S = \int_c^d 2\pi y g(y) \, dy.$$ 

Example

Find the volume of the solid obtained by rotating about the $x$-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

radius = $y$, circumference = $2\pi y$, height = $1 - y^2$

volume = $\int_0^1 (2\pi y)(1 - y^2) \, dy$ (The cross-section method seems better.)
Example

Find the volume of the solid obtained by rotating the region under the curve $y = x - x^2$ and $y = 0$ about **the line** $x = 2$.

radius = $2 - x$, circumference = $2\pi(2 - x)$, height = $x - x^2$

$$\text{volume} = \int_{0}^{1} 2\pi(2 - x)(x - x^2) \, dx$$
6-4 Work

Definition
An object moves along the $x$-axis in the positive direction. At each point $x$ a force $f(x)$ acts continuously at the object. The work done in moving the object from $x = a$ to $x = b$ is

$$\text{work} = \int_{a}^{b} f(x) \, dx.$$ 

Example
A force of 40 (newton) is needed to hold a spring that has been stretched from its natural length of 10 (cm) to a length of 15. How much work is done in stretching the spring from 15 to 18?

$$f(x) = kx \quad \text{Hooke's Law}$$
6-4 Work

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An object moves along the $x$-axis in the positive direction. At each point $x$ a force $f(x)$ acts continuously at the object. The work done in moving the object from $x = a$ to $x = b$ is

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$$f(x) = kx \quad \text{Hooke’s Law}$$
6-4 Work

Example

A 200 (lb) cable is 100 (ft) long and hangs vertically from the top of a building. Set the top of the building to be the origin and the $x$-axis pointing downward. Partition the cable into $n$ small pieces of uniform length $\Delta x$, and let $x_i^*$ denote a point in the $i$th such small piece. Assume the cable is made of uniform density so that it weighs 2 per foot (lb/ft), so the weight of the $i$th part is $2\Delta x$ (lb).

\[
\text{work done on the } i\text{th part} = \left( 2\Delta x \right) \cdot \left( x_i^* \right)
\]

force against gravity  \hspace{1cm} distance

Overall, the work is needed to lift the cable to the top of the building is given by

\[
\lim_{n \to \infty} \sum_{i=1}^{n} 2x_i^* \Delta x = \int_{0}^{100} 2x \, dx.
\]
6-4 Work

Example

A water tank has the shape of an inverted circular cone with height 10 (m) and base radius 4 (m). It is filled with water to a height of 8 (m). Find the work required to empty tank by pumping all of the water to the top of the tank. (The density of water is 1000.)

Hint: Measure depth from the top of the tank by placing $x = 0$ at there, and partition the (vertical) interval $[2, 10]$ into $n$ subintervals and choose $x_i^*$ from the $i$-th one, so that the water is divided into $n$ layers. Then the $i$-th layer is approximated by a circular cylinder with radius $r_i$ and height $\Delta x = 8/n$, where

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10}.$$

Note that the density of water is 1000 (kg/m$^3$), the gravitational constant $g = 9.8$, and mass = density $\times$ volume.
6-5 Average Value of a Function

**Definition**
The average value of a function \( f \) on \([a, b]\) is

\[
\frac{1}{b-a} \int_{a}^{b} f(x) \, dx.
\]

**Theorem**
*(Mean Value Theorem)* If \( f \) is continuous on \([a, b]\), then there exists a number \( c \) in \([a, b]\) such that

\[
f(c) = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx.
\]

**Example**
Find such a \( c \) for \( f(x) = 1 + x^2 \) on \([-1, 2]\).
6-5 Average Value of a Function

**Definition**

The average value of a function $f$ on $[a, b]$ is

$$\frac{1}{b-a} \int_a^b f(x) \, dx.$$ 

**Theorem**

*(Mean Value Theorem)* If $f$ is continuous on $[a, b]$, then there exists a number $c$ in $[a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx.$$ 

**Example**

Find such a $c$ for $f(x) = 1 + x^2$ on $[-1, 2]$. 

7-1 Integration by Parts

Definition
(Integration by Parts)
\[ \int u \, dv = uv - \int v \, du. \]

Example
Find \( \int x \sin x \, dx \).

Example
Find \( \int \ln x \, dx \).

Example
Find \( \int t^2 e^t \, dt \).

Example
Find \( \int e^x \sin x \, dx \).
7-1 Integration by Parts

Definition
(Integration by Parts)
\[ \int u \, dv = uv - \int v \, du. \]

Example
Find \( \int_{0}^{1} \tan^{-1} x \, dx \). \( u = \tan^{-1} x, \ dv = dx \)

Example
\[ \int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx. \]

Example
\[ \int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx. \]
Example

Find \( \int \cos^3 x \, dx \). \( \cos^2 x + \sin^2 x = 1 \)

Example

Find \( \int \sin^5 x \cos^2 x \, dx \).

Example

Find \( \int_0^\pi \sin^2 x \, dx \). \( \sin^2 x = (1 - \cos 2x)/2 \)

Example

Find \( \int \sin^4 x \, dx \). Use the reduction formula.
Fact

To evaluate $\int \sin^m x \cos^n x \, dx$:

1. If $n$ is odd, separate one $\cos x$ out and use $\cos^2 x = 1 - \sin^2 x$.
2. If $m$ is odd, separate one $\sin x$ out and use $\cos^2 x = 1 - \sin^2 x$.
3. If both $m, n$ are even, use

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin x \cos x = \frac{\sin 2x}{2}.$$

Example

$\int \sin^4 x \cos^4 x \, dx$
Fact

To evaluate $\int \sin^m x \cos^n x \, dx$:

1. If $n$ is odd, separate one $\cos x$ out and use $\cos^2 x = 1 - \sin^2 x$.
2. If $m$ is odd, separate one $\sin x$ out and use $\cos^2 x = 1 - \sin^2 x$.
3. If both $m, n$ are even, use
   
   \[ \sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin x \cos x = \frac{\sin 2x}{2}. \]

Example

$\int \sin^4 x \cos^4 x \, dx$
Fact

To evaluate $\int \sin^m x \cos^n x \, dx$:

1. If $n$ is odd, separate one $\cos x$ out and use $\cos^2 x = 1 - \sin^2 x$.
2. If $m$ is odd, separate one $\sin x$ out and use $\cos^2 x = 1 - \sin^2 x$.
3. If both $m, n$ are even, use

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin x \cos x = \frac{\sin 2x}{2}.$$  

Example

$$\int \sin^4 x \cos^4 x \, dx$$
7-2 Trig Integrals

**Fact**

To evaluate $\int \sin^m x \cos^n x \, dx$:

1. If $n$ is odd, separate one $\cos x$ out and use $\cos^2 x = 1 - \sin^2 x$.
2. If $m$ is odd, separate one $\sin x$ out and use $\cos^2 x = 1 - \sin^2 x$.
3. If both $m, n$ are even, use

   $$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin x \cos x = \frac{\sin 2x}{2}.$$

**Example**

$$\int \sin^4 x \cos^4 x \, dx$$
### 7-2 Trig Integrals

**Example**

Find \( \int \tan^6 x \sec^4 x \, dx \).  \( \sec^2 x = 1 + \tan^2 x \),  \( u = \tan x \),  \( du = \sec^2 x \, dx \)

**Example**

Find \( \int \tan^5 x \sec^7 x \, dx \).  \( u = \sec x \),  \( du = \sec x \tan x \, dx \)

**Example**

Find \( \int \tan x \, dx = \ln |\sec x| + C \),  \( \tan x = \frac{\sin x}{\cos x} \)

**Example**

Find \( \int \sec x \, dx = \ln |\sec x + \tan x| + C \)
7-2 Trig Integrals

Fact

To evaluate \( \int \tan^m x \sec^n x \, dx \):

1. If \( n \) is even, separate one \( \sec^2 x \) out and use \( \sec^2 x = 1 + \tan^2 x \).
2. If \( m \) is odd, separate one \( \sec x \tan x \) out and use \( \tan^2 x = \sec^2 x - 1 \).

Example

\( \int \tan^3 x \, dx \); use \( \tan^2 x = \sec^2 x - 1 \) first then follow (1).

Example

\( \int \sec^3 x \, dx \); use integration by parts: \( u = \sec x, \ dv = \sec^2 x \, dx \)

\[ - \int \tan^2 x \sec x \, dx = - \int (\sec^2 x - 1) \sec x \, dx = " - " \int \sec^3 x \, dx + \int \sec x \, dx \]
7-2 Trig Integrals

Fact

To evaluate \( \int \tan^m x \sec^n x \, dx \):

1. If \( n \) is even, separate one \( \sec^2 x \) out and use \( \sec^2 x = 1 + \tan^2 x \).
2. If \( m \) is odd, separate one \( \sec x \tan x \) out and use \( \tan^2 x = \sec^2 x - 1 \).

Example

\( \int \tan^3 x \, dx \); use \( \tan^2 x = \sec^2 x - 1 \) first then follow (1).

Example

\( \int \sec^3 x \, dx \); use integration by parts: \( u = \sec x \), \( dv = \sec^2 x \, dx \)

\( - \int \tan^2 x \sec x \, dx = - \int (\sec^2 x - 1) \sec x \, dx = " - " \int \sec^3 x \, dx + \int \sec x \, dx \)
7-2 Trig Integrals

Fact

To evaluate $\int \tan^m x \sec^n x \, dx$:

1. If $n$ is even, separate one $\sec^2 x$ out and use $\sec^2 x = 1 + \tan^2 x$.
2. If $m$ is odd, separate one $\sec x \tan x$ out and use $\tan^2 x = \sec^2 x - 1$.

Example

$\int \tan^3 x \, dx$; use $\tan^2 x = \sec^2 x - 1$ first then follow (1).

Example

$\int \sec^3 x \, dx$; use integration by parts: $u = \sec x$, $dv = \sec^2 x \, dx$

\[- \int \tan^2 x \sec x \, dx = - \int (\sec^2 x - 1) \sec x \, dx = "-" \int \sec^3 x \, dx + \int \sec x \, dx\]
7-2 Trig Integrals

Fact

**Product-Sum Formulas**

1. \( \sin A \cos B = \frac{1}{2} \left[ \sin(A - B) + \sin(A + B) \right] \)
2. \( \sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \)
3. \( \cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)] \)

Example

\( \int \sin 4x \cos 5x \, dx \)
7-2 Trig Integrals

Fact

Product-Sum Formulas

1. \( \sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)] \)
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Example

\[ \int \sin 4x \cos 5x \, dx \]
### 7-3 Trig Substitution

**Fact**

**Table of Trig Substitutions**

1. \( \sqrt{a^2 - x^2} \Rightarrow x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 1 - \sin^2 \theta = \cos^2 \theta. \)
2. \( \sqrt{a^2 + x^2} \Rightarrow x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad 1 + \tan^2 \theta = \sec^2 \theta. \)
3. \( \sqrt{x^2 - a^2} \Rightarrow x = a \sec \theta, \quad 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}, \quad \sec^2 \theta - 1 = \tan^2 \theta. \)

**Example**

\( \int \frac{\sqrt{9-x^2}}{x^2} \, dx, \quad \int \frac{1}{x^2\sqrt{x^2+4}} \, dx, \quad \int \frac{x}{\sqrt{x^2+4}} \, dx, \quad \int \frac{1}{\sqrt{x^2-a^2}} \, dx. \)
Fact

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Example

\[
\int \frac{\sqrt{9-x^2}}{x^2} \, dx, \quad \int \frac{1}{x^2 \sqrt{x^2+4}} \, dx, \quad \int \frac{x}{\sqrt{x^2+4}} \, dx, \quad \int \frac{1}{\sqrt{x^2-a^2}} \, dx.
\]
7-3 Trig Substitution

Fact

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Example

\[ \int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} \, dx, \quad \int \frac{x}{\sqrt{3-2x-x^2}} \, dx \]
7-4 Partial Fractions (Long Division Reduction)

**Fact**

*Long Division Algorithm*

\[
\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}
\]

**Example**

\[
\int \frac{x^3 + x}{x - 1} \, dx
\]
Fact

If the denominator \( g(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k) \) where no factor is repeated, then

\[
\frac{f(x)}{g(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}
\]

Example

\[
\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \, dx
\]

\[
\int \frac{1}{x^2 - a^2} \, dx
\]
Fact

If some factors are repeated, say, \((a_1x + b_1)^r\), then one replaces

\[
\frac{A_1}{a_1x + b_1}
\]

by

\[
\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r},
\]

and do this for each repeated factor.

Example

\[
\frac{x^3 - x + 1}{x^2(x - 1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2} + \frac{E}{(x - 1)^3}
\]

\[
\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \, dx
\]
7-4 Partial Fractions (Distinct Irreducible Quadratic Factors)

Fact

If there is an irreducible quadratic factor \( ax^2 + bx + c \), then in addition to the partial fractions in previous cases, one adds

\[
\frac{Ax + B}{ax^2 + bx + c}.
\]

Example

\[
\frac{x}{(x - 2)(x^2 + 1)(x^2 + 4)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{x^2 + 4}
\]

Note that

\[
\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C.
\]
7-4 Partial Fractions (Repeated Irreducible Quadratic Factors)

**Fact**

*If there is a repeated irreducible quadratic factor \((ax^2 + bx + c)^r\), then in addition to the partial fractions in previous cases, one replaces*

\[
\frac{Ax + B}{ax^2 + bx + c}
\]

*by*

\[
\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}.
\]

**Example**

\[
\frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}
\]

\[
\int \frac{\sqrt{x + 4}}{x} \, dx; \; u = \sqrt{x + 4}
\]
7-5 Integration Strategy

Example

\[ \int \frac{\tan^3 x}{\cos^3 x} \, dx; \quad \frac{\tan^3 x}{\cos^3 x} = \tan^3 x \sec^3 x = \frac{\sin^3 x}{\cos^6 x} \]
\[
\begin{align*}
&= \sec x \quad u = \sec x \\
&= \cos x \quad u = \cos x
\end{align*}
\]

Example

\[ \int e^{\sqrt{x}} \, dx = 2 \int ue^u \, du \]

Example

\[ \int \frac{1}{x\sqrt{\ln x}} \, dx = \int \frac{1}{\sqrt{u}} \, du; \quad \int \sqrt{\frac{1-x}{1+x}} \, dx, \quad \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C 
\]

\[
\begin{align*}
&= \sqrt{u} \\
&= u
\end{align*}
\]
7-5 Integration Strategy

Example

\[
\int \frac{\tan^3 x}{\cos^3 x} \, dx; \quad \frac{\tan^3 x}{\cos^3 x} = \tan^3 x \sec^3 x = \frac{\sin^3 x}{\cos^6 x}
\]

\(u = \sec x\)

\(u = \cos x\)

Example

\[
\int e^{\sqrt{x}} \, dx = 2 \int u e^u \, du
\]

Example

\[
\int \frac{1}{x \sqrt{\ln x}} \, dx = \int \frac{1}{\sqrt{u}} \, du; \quad \int \sqrt{\frac{1-x}{1+x}} \, dx, \quad \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C
\]

\(u = u\)

\(u = \sin^{-1} x + C\)

7-5 Integration Strategy

Example

\[
\int \frac{\tan^3 x}{\cos^3 x} \, dx; \quad \tan^3 x = \tan^3 x \sec^3 x = \frac{\sin^3 x}{\cos^6 x}
\]

\(u = \sec x\)

\(u = \cos x\)

Example

\[
\int e^{\sqrt{x}} \, dx = 2 \int ue^u \, du
\]

Example

\[
\int \frac{1}{x \sqrt{\ln x}} \, dx = \int \frac{1}{\sqrt{u}} \, du; \quad \int \sqrt{\frac{1-x}{1+x}} \, dx, \quad \int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C
\]

\(u = \sqrt{\ln x}\)

\(u = \sqrt{\frac{1-x}{1+x}}\)

\(u = \sqrt{1-x^2}\)
7-7 Approximation Integration

Fact

\[
\int_{a}^{b} f(x) \, dx \approx \sum_{i=1}^{n} f(x_{i-1}) \Delta x \quad \text{(left point approximation)}
\]

\[
\int_{a}^{b} f(x) \, dx \approx \sum_{i=1}^{n} f(x_{i}) \Delta x \quad \text{(right point approximation)}
\]

\[
\int_{a}^{b} f(x) \, dx \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_{i}}{2}\right) \Delta x \quad \text{(midpoint approximation)}
\]
7-7 Approximation Integration

Fact

\[ \int_a^b f(x) \, dx \approx \sum_{i=1}^{n} f(x_{i-1}) \Delta x \quad \text{(left point approximation)} \]

\[ \int_a^b f(x) \, dx \approx \sum_{i=1}^{n} f(x_i) \Delta x \quad \text{(right point approximation)} \]

\[ \int_a^b f(x) \, dx \approx \sum_{i=1}^{n} \left[ \frac{f(x_{i-1}) + f(x_i)}{2} \right] \Delta x \quad \text{(Trapezoidal Rule)} \]

Trapezoidal = average of the left and the right
7-7 Approximation Integration

**Definition**

The **error** in an approximation is defined to the

the error = the exact value − the approximation.

**Example**

The Trapezoidal Rule for $\int_{1}^{2} \frac{1}{x} \, dx$ where $n = 5$ gives the approximation

$$T = 0.695635,$$

then

$$\text{error} = \int_{1}^{2} \frac{1}{x} \, dx - T = \ln 2 - T = -0.002488.$$
7-7 Approximation Integration

**Theorem**

*(Error Bounds)* Suppose

\[ |f''(x)| \leq K \quad \text{for } a \leq x \leq b. \]

Then

\[ |E_T| \leq \frac{K(b - a)^3}{12n^2}, \quad |E_M| \leq \frac{K(b - a)^3}{24n^2}, \]

where \( E_T \) and \( E_M \) denote respectively the errors in the Trapezoidal and Midpoint Rules.

**Example**

The Trapezoidal Rule of \( \int_1^2 \frac{1}{x} \, dx \) for \( n = 5 \) yields

\[ |E_T| \leq \frac{2(2 - 1)^3}{12 \cdot 5^2}. \]
Theorem

(Error Bounds) Suppose

\[ |f''(x)| \leq K \text{ for } a \leq x \leq b. \]

Then

\[ |E_T| \leq \frac{K(b-a)^3}{12n^2}, \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}, \]

where \( E_T \) and \( E_M \) denote respectively the errors in the Trapezoidal and Midpoint Rules.

Example

How large should we take \( n \) in order to guarantee that the Trapezoidal approximation for \( \int_1^2 \frac{1}{x} \, dx \) is accurate within 0.0001?

\[ |E_T| \leq \frac{2(2-1)^3}{12 \cdot n^2} < 0.0001 \]
Theorem

(Simpson’s Rule) Assume \( n \) is an even number.

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right],
\]

where \( n \) is even and \( \Delta x = (b - a)/n \).

pattern = 1, 4, 2, 4, 2, \cdots , 4, 2, 4, 1

Theorem

(Error Bound) Suppose

\[ |f^{(4)}(x)| \leq K, \quad a \leq x \leq b. \]

Then

\[ |E_s| \leq \frac{K(b - a)^5}{180n^4}. \]
4-4 L’Hospital’s Rule

**Theorem**

Let \( f \) and \( g \) be differentiable on an open interval \( I \) containing \( a \). Suppose that

\[
\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x),
\]

or that

\[
\lim_{x \to a} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm \infty.
\]

Then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},
\]

provided that the limit on the right side exists (or is \( \infty \) or \( -\infty \)).

**Examples**

\[
\lim_{x \to 1} \frac{\ln x}{x - 1}, \quad \lim_{x \to \infty} \frac{e^x}{x^2}, \quad \lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}, \quad \lim_{x \to 0} \frac{\tan x - x}{x^3}
\]
Theorem

Let $f$ and $g$ be differentiable on an open interval $I$ containing $a$. Suppose that

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x),$$

or that

$$\lim_{x \to a} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm \infty.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right side exists (or is $\infty$ or $-\infty$).

Examples

$$\lim_{x \to 0^+} x \ln x, \quad \lim_{x \to 1^+} \left( \frac{1}{\ln x} - \frac{1}{x-1} \right), \quad \lim_{x \to \infty} (e^x - x)$$
4-4 L’Hospital’s Rule

Theorem

Let $f$ and $g$ be differentiable on an open interval $I$ containing $a$. Suppose that

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x),$$

or that

$$\lim_{x \to a} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm \infty.$$  

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right side exists (or is $\infty$ or $-\infty$).

Examples

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}, \quad \lim_{x \to 0^+} x^x$$
7-8 Improper Integrals

Definitions

Improper integrals of type 1 (over unbounded intervals):

1. \[ \int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx \]

2. \[ \int_{-\infty}^b f(x) \, dx = \lim_{t \to \infty} \int_t^b f(x) \, dx \]

3. \[ \int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^a f(x) \, dx + \int_a^\infty f(x) \, dx \]

Example

Determine whether the (improper) integral \( \int_1^\infty \frac{1}{x} \, dx \) is convergent or divergent. More generally, how about \( \int_1^\infty \frac{1}{x^p} \, dx \) for \( p > 0 \)?
7-8 Improper Integrals

## Definitions

Improper integrals of type 2 (with discontinuities):

1. **$f$ is discontinuous at $b$:**

   \[
   \int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx
   \]

2. **$f$ is discontinuous at $a$:**

   \[
   \int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx
   \]

3. **$f$ is discontinuous at $c$, where $a < c < b$:**

   \[
   \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
   \]
7-8 Improper Integrals

Theorem

(Comparison Test) If \(0 \leq g \leq f\), then

1. If \(\int_{a}^{\infty} f(x) \, dx\) is convergent, then \(\int_{a}^{\infty} g(x) \, dx\) is convergent.

2. If \(\int_{a}^{\infty} g(x) \, dx\) is divergent, then \(\int_{a}^{\infty} f(x) \, dx\) is divergent.

Example

\(\int_{0}^{\infty} e^{-x^2} \, dx\) is convergent and \(\int_{1}^{\infty} \frac{1+e^{-x}}{x} \, dx\) is divergent.

Example

\(\int_{2}^{5} \frac{1}{\sqrt{x-2}} \, dx\), \(\int_{0}^{\pi/2} \sec x \, dx\), \(\int_{0}^{3} \frac{1}{x-1} \, dx\), \(\int_{0}^{1} \ln x \, dx\)
8-1 Arc Length

Definition
If $f'$ is continuous on $[a, b]$ then the length of the curve $y = f(x)$, $a \leq x \leq b$, is given by

$$L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx.$$  

Using Leibniz notation, the formula can be rewritten as

$$L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$  

Example
Find $L$ for $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.  

8-1 Arc Length

Definition

If \( g'(y) \) is continuous on \([c, d]\) then the length of the curve \( x = g(y) \), \( c \leq y \leq d \), is given by

\[
L = \int_c^d \sqrt{1 + [g'(y)]^2} \, dy.
\]

Using Leibniz notation, the formula can be rewritten as

\[
L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy.
\]

Example

Find \( L \) for \( y^2 = x \) between the points \((0,0)\) and \((1,1)\).
8-1 Arc Length

Definition

Given a curve \( y = f(x), \ a \leq x \leq b \), let

\[
s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} \, dt, \quad a \leq x \leq b,
\]

be the arc length from point \((a, f(a))\) to \((x, f(x))\). \(s(x)\) is called the arc length function. Note that FTC implies that

\[
s'(x) = \sqrt{1 + [f'(x)]^2}, \quad \text{or equivalently,}
\]

\[
ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.
\]

Example

Find \( s(x) \) for \( y = x^2 - \frac{1}{8} \ln x \) starting at the point \((1, 1)\).
8-2 Area of a Surface of Revolution

**Definition**

Consider the surface obtained by rotating the curve $y = f(x) \geq 0$, $a \leq x \leq b$, about the $x$-axis, the surface area is given by

$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx = \int_a^b 2\pi y \sqrt{1 + \left[\frac{dy}{dx}\right]^2} \, dx = \int_a^b 2\pi y ds.$$  

For rotation about $y$-axis of $x = g(y)$, $c \leq y \leq d$, we have

$$S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} \, dy = \int_c^d 2\pi x \sqrt{1 + \left[\frac{dx}{dy}\right]^2} \, dy = \int_c^d 2\pi x ds.$$  

**Example**

Find $S$ when rotating $y = \sqrt{4-x^2}$, $-1 \leq x \leq 1$, about $x$-axis.
8-2 Area of a Surface of Revolution

**Definition**

Consider the surface obtained by rotating the curve \( y = f(x) \geq 0, \ a \leq x \leq b \), about the \( x \)-axis, the surface area is given by

\[
S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx = \int_a^b 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_a^b 2\pi y \, ds.
\]

For rotation about \( y \)-axis of \( x = g(y) \), \( c \leq y \leq d \), we have

\[
S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} \, dy = \int_c^d 2\pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = \int_c^d 2\pi x \, ds.
\]

**Example**

Find \( S \) when rotating \( y = e^x, \ 0 \leq x \leq 1 \), about \( x \)-axis.
3-8 Exponential Growth and Decay

### Definition

If $y(t)$ is the value of a quantity $y$ at the time $t$ and if the rate of change of $y$ with respect to $t$ is proportional to its size $y(t)$ at any time $t$, then

$$\frac{dy}{dt} = ky \quad \text{for some constant } k,$$

and the only solution for this differential equation is

$$y(t) = y(0)e^{kt}.$$

The constant $k$ is called the **relative growth rate** of the quantity $y$.

### Example

Suppose the growth rate of a certain population is proportional to the population size $P(t)$, and say, $P(0) = 2560$ and $P(10) = 3040$. Then the relative growth rate is $k = 0.017$ and $P(t) = 2560e^{kt}$. 
Example

The half-life of a certain radioactive element is 1590 years.

1. Find a formula for the mass $m(t)$ of the element that remains after $t$ years. Suppose $m(0) = 100$.

2. Find the mass $m(1000)$ after 1000 years.

3. When will the mass be reduced to 30?

Example

Newton’s law of cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s),$$

where $k$ is a constant and $T_s$ is the (constant) temperature of surroundings. Make a change of variable $y(t) = T(t) - T_s$ to rewrite it as $y' = ky$. 
Example

The half-life of a certain radioactive element is 1590 years.

1. Find a formula for the mass $m(t)$ of the element that remains after $t$ years. Suppose $m(0) = 100$.
2. Find the mass $m(1000)$ after 1000 years.
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Example

Newton’s law of cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s),$$

where $k$ is a constant and $T_s$ is the (constant) temperature of surroundings. Make a change of variable $y(t) = T(t) - T_s$ to rewrite it as $y' = ky$. 

Example

Denote by \( A(t) \) the amount of a financial investment at time \( t \). The continuous compounding of \( A \) with interest rate \( r \) is governed by the differential equation:

\[
\frac{dA}{dt} = rA(t).
\]

For example, $1000 invested for 3 years at 6% interest rate will have its value

\[
A(3) = 1000e^{(0.06)3} = 1197.22.
\]
Example

The equation

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{M}\right)
\]

shows that

1. If \( P \) is small, then

\[
\frac{dP}{dt} \approx kP. \text{ (Initially, the growth rate is proportional to } P. \text{)}
\]

2. If \( P > M \), then

\[
\frac{dP}{dt} < 0. \text{ (} P \text{ decreases if it ever exceeds the constant } M. \text{)}
\]
Example

Show that every member of the family of functions

\[ y = \frac{1 + ce^t}{1 - ce^t}, \quad c \text{ is any constant}, \]

satisfies the differential equation

\[ y' = \frac{1}{2} (y^2 - 1). \]

Moreover, the solution of the equation \( y' = \frac{1}{2} (y^2 - 1) \) satisfying the initial condition \( y(0) = 2 \) is

\[ y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t}. \]
9-3 Separable Equations

**Definition**

\[ \frac{dy}{dx} = g(x)f(y) \]

**Example**

\[ y' = \frac{x^2}{y^2}, \quad y(0) = 2. \]

**Example**

\[ y' = \frac{6x^2}{2y + \cos y} \]

**Example**

\[ y' = x^2y \]
Example

A water tank contains 20 kg of salt dissolved in 5000 L of water. Salted water that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/minute. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after 30 minutes?

\[ y(t) = \text{amount of salt at time } t \]
\[ y'(t) = \text{(rate in)} - \text{(rate out)}, \quad y(0) = 20. \]

rate in \[= 0.03 \frac{kg}{L} \times 25 \frac{L}{min} = 0.75 \frac{kg}{min} \]

rate out \[= \frac{y(t)}{5000} \frac{kg}{L} \times 25 \frac{L}{min} = \frac{y(t)}{200} \frac{kg}{min} \]
To be continued
To be continued
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Examples

Example
Examples

Example